Optimal reinsurance minimizing the distortion risk measure under general reinsurance premium principles

Wei Cui∗, Jingping Yang†, Lan Wu‡

November 13, 2011

Abstract

Recently the optimal reinsurance strategy concerning the insurer’s risk attitude and the reinsurance premium principle is an interesting topic. This paper discusses the optimal reinsurance problem with the insurer’s risk measured by distortion risk measure and the reinsurance premium calculated by a general principle including expected premium principle and Wang’s premium principle as its special cases. Explicit solutions of the optimal reinsurance strategy are obtained under the assumption that both the ceded loss and the retained loss are increasing with the initial loss. We present a new method for discussing the optimal problem. Based on our method, one can explain the optimal reinsurance treaty in the view of a balance between the insurer’s risk measure and the reinsurance premium principle.

Key-words: Optimal reinsurance; Distortion risk measure; Reinsurance premium principle; VaR; TVaR.

∗Department of Financial Mathematics, Peking University, Beijing, 100871
†Department of Financial Mathematics, Center for Statistical Science, Peking University, Beijing, 100871, China. Email: yangjp@math.pku.edu.cn
‡Department of Financial Mathematics, Peking University, Beijing, 100871, China. Email: lwu@pku.edu.cn
1 Introduction

Reinsurance is a risk management tool for an insurance company. By balancing paid loss and reinsurance premium, an insurer can control its risk by ceding part of its risk to a reinsurer. The optimality of the reinsurance methodology can be explained through mathematical models. For one policy with the initial loss $X$, it is normally assumed that $X \geq 0$ and $0 < E[X] < \infty$. The insurer cedes part of its loss, denoted by $f(X)$, to a reinsurer. In return, the reinsurer will receive a reinsurance premium, denoted as $\mu(f(X))$. For the agreement, the insurer’s paid loss can be expressed as $I_f(X) = X - f(X)$. The total payment of the insurer, denoted as $T_f(X)$, can be expressed as the sum of the paid loss $I_f(X)$ and the reinsurance premium $\mu(f(X))$.

$$T_f(X) = I_f(X) + \mu(f(X)).$$

The normal reinsurance strategies include stop-loss reinsurance with reinsurer’s loss payment $\max\{X - d, 0\}, d > 0$, quota-share reinsurance with reinsurer’s loss payment $(1 - \alpha)X, 0 < \alpha < 1$, limited stop-loss reinsurance with reinsurer’s loss payment $\min\{\max\{X - d_1, 0\}, d_2 - d_1\}, d_2 > d_1 > 0$, and the mixture of the above forms.

From quantitative point, the insurer’s risk can be measured by some risk measure with desirable properties, and the reinsurance premium can be determined under some premium principle. The optimal reinsurance strategy can be derived by minimizing the insurer’s risk measure under the reinsurance premium principle. The research on risk measure minimization reinsurance models can retrospect to Borch (1960). Borch (1960) used variance as the insurer’s risk measure and found that stop-loss reinsurance is optimal under the expected premium principle. After Borch (1960), a number of results have been obtained in this so-called variance minimization models with more realistic premium principles, see Gajeck and Zagrodny (2000) and Kaluszka (2001). In recent years, minimizing some general risk measures is a hot topic, see Kaluszka.
(2004a), Gajeck et al (2004a), Kaluszka(2005), and Balbâs et al. (2009). Moreover, motivated by the capital requirement of Basel II agreement, some studies on optimal reinsurance focus on Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) as their risk measures. With the reinsurance premium calculated by expected premium principle, by minimizing either VaR or TVaR of the insurer’s total risk, Cai et al. (2008) showed that stop-loss reinsurance is optimal when the ceded loss functions are convex and increasing, and Cheung (2010) obtained the same results by a geometric approach. Cheung (2010) also discussed the VaR optimization problem under Wang’s premium principle (Wang, Young and Panjer, 1997). Tan, Weng and Zhang (2011) studied TVaR optimization problem under an enlarged class of ceded loss functions. Chi and Tan (2010) and Chi and Tan (2011) investigated the VaR and TVaR optimization reinsurance models by focusing on different classes of ceded loss functions and different premium principles. The optimal reinsurance strategy can also be investigated by using other methodology, see Arrow (1963), Young (1999) and Kaluszka (2008) for expected utility maximization reinsurance models, and Gajek (2004b), Kaluszka and Okolewski (2008) and Bernard and Tian (2009) for ruin probability minimization reinsurance models.

The distortion risk measure (Dhaene et al., 2006) is a popular risk measure that is widely used. A distortion function \( g(x) : [0,1] \rightarrow [0,1] \) is an increasing function satisfying that \( g(0) = 0 \) and \( g(1) = 1 \). The distortion risk measure associated with the distortion function \( g \) is defined as

\[
\rho_g[X] = \int_0^\infty g(P(X > x))dx.
\]  

(1)

Many popular risk measures belong to the family of distortion risk measures, such as VaR, TVaR and Wang’s transform risk measure, corresponding to the distortion functions \( g(x) = I_{\{x>\alpha\}}, g(x) = \min\{\frac{x}{\alpha}, 1\} \) and \( g_\alpha(x) = \Phi[\Phi^{-1}(x) + \Phi^{-1}(\alpha)] \), respectively. In this paper, we will use the distortion risk measure to measure the insurer’s risk.
We assume that the reinsurance premium is determined by the following premium principle:

$$\mu_r(f(X)) = \int_0^\infty r(P(f(X) > s))ds.$$  

(2)

Here $r(x), x \in [0, 1]$, is a bounded left-continuous and increasing function with $r(0) = 0$ and $r(x) > 0$ for all $x > 0$. Note that the expected premium principle and Wang’s premium principle belong to the above family. If we assume that for $\beta > 0$, $r(x) = (1 + \beta)x, x \geq 0$ in (2), it is the expected premium principle with safety loading $\beta > 0$. When $r(x)$ is an increasing concave function with $r(0) = 0, r(1) = 1$, it becomes Wang’s premium principle.

In this paper we make the following assumptions:

- **Insurer’s risk measure.** The insurer’s risk is measured by the distortion risk measure in (1), and for technique reason we assume that $g$ is left-continuous;

- **Reinsurance premium principle.** The insurer pays reinsurance premium according to the premium principle $\mu_r(f(X))$ in (2);

- **The class of ceded loss function.** The ceded loss function $f(x)$ satisfies that $0 \leq f(x) \leq x, x \geq 0$, and that the two functions $f(x)$ and $I_f(x) = x - f(x)$ are increasing functions. The class of total $f$’s satisfying the above assumption is denoted as $\mathcal{F}$.

Note that the increasing assumption on the function $x - f(x)$ is natural. It guarantees that there will be no moral hazard, since both the insurer and reinsurer will pay more when the initial loss increases. When $f \in \mathcal{F}$, the paid loss of the insurer and the paid loss of the reinsurer are comonotonic, and the two functions $f(x)$ and $I_f(x) = x - f(x)$ are continuous.

Under the above assumptions, this paper will focus on the following two optimization problems:
• The optimization problem without premium constraint
\[
\min_{f \in \mathcal{F}} \{ \rho_g[T_f(X)] \}; \quad (3)
\]

• The optimization problem with premium constraint
\[
\begin{cases}
\min_{f \in \mathcal{F}} \{ \rho_g[T_f(X)] \} \\
\text{s.t. } \mu_r(f(X)) \leq \pi,
\end{cases} \quad (4)
\]

where the constant \( \pi \) is the largest amount the insurer would like to pay for the reinsurance premium.

By limiting the feasible ceded loss function within the class \( \mathcal{F} \), we will find the explicit solutions for the reinsurance optimization models (3) and (4).

We can explain our models from the following four aspects:

• Our premium principle include some important ones, such as expected premium principle and Wang’s premium principle;

• The class of distortion risk measures assumed in this paper include some widely used measures, such as VaR and TVaR. Note that VaR and TVaR optimization models are discussed in Cai et al. (2008), Cheung (2010), Chi and Tan (2010), Chi and Tan (2011) and Tan, Weng and Zhang (2011);

• The optimal reinsurance strategies obtained in this paper have forms of layer reinsurance. The insurer can partition the range of the claim into several excess-of-loss layers, and on different layers the insurer may adopt distinct reinsurance strategies. Some clinical evidences unfold that many reinsurance contracts, especially those contain catastrophe risk, such as the USAA reinsurance program in Froot (2001), have such layer reinsurance form. As far as we know, most optimal reinsurance designs in the former papers have only one layer, such as quota share, stop-loss and limited stop-loss, see Chi and Tan (2011) for details;
In this paper we present a new method for discussing the optimal problems. The method can illustrate the essence of how to arrange the optimal reinsurance. For the optimal reinsurance treaty of each loss $X$, the boundaries of each layer and the corresponding reinsurance arrangement on the layer can be set up by comparing the two functions $r(P(X > t)), t \geq 0$ and $g(P(X > t)), t \geq 0$. On the layer where the premium function $r(P(X > t))$ is less than the risk measure function $g(P(X > t))$, buying a full coverage on the excess loss of the layer will benefit the insurer most. On the other-hand, for the layers where $r(P(X > t)) > g(P(X > t))$, retaining all those incremental losses will be optimal.

The paper is organized as follows. In section 2, explicit solutions for our optimal reinsurance models are given. In section 3, we analyze the influence of the expected premium principle and Wang’s premium principle on the optimal reinsurance by comparing with the existing results. The proofs of our main theorems are provided in section 4. In section 5 we draw our conclusions.

2 Main Results on the optimal reinsurance

2.1 Preparation work

In our optimal reinsurance models, the distortion risk measure and the reinsurance premium principle will determine the form of the optimal ceded loss function. In the following, one fundamental method will be provided for solving the optimal problems. By comparing the function $r(x), x \in [0,1]$ in the premium principle (2) and the distortion function $g(x), x \in [0,1]$ in the distortion risk measure (1), the interval $[0,1]$ can be partitioned into subintervals according to the relationship $r(x) < g(x), r(x) = g(x)$ and $r(x) > g(x)$, respectively. The partition will be applied for constructing the optimal reinsurance in the following sections.
For simplicity, we denote
\[
\Omega_{r,g}^{-} := \{0 \leq x \leq 1 : r(x) - g(x) < 0\}, \quad \Omega_{r,g}^{0} := \{0 \leq x \leq 1 : r(x) - g(x) = 0\}
\]
and
\[
\Omega_{r,g}^{+} := \{0 \leq x \leq 1 : r(x) - g(x) > 0\}.
\]
Thus we have
\[
\Omega_{r,g}^{-} \cup \Omega_{r,g}^{0} \cup \Omega_{r,g}^{+} = [0, 1].
\]
Since \(r(0) = g(0) = 0\), then \(0 \in \Omega_{r,g}^{0}\).

**Proposition 2.1.** We have the following partitions
\[
\Omega_{r,g}^{-} = \bigcup_{j=1}^{n_{r,g}^{(1)}} (q_{r,g}^{-}(j), q_{r,g}^{+}(j)) \bigcup_{j=1}^{n_{r,g}^{(2)}} (\hat{q}_{r,g}^{-}(j), \hat{q}_{r,g}^{+}(j)), \quad (5)
\]
\[
\Omega_{r,g}^{+} = \bigcup_{j=1}^{k_{r,g}^{(1)}} (s_{r,g}^{-}(j), s_{r,g}^{+}(j)) \bigcup_{j=1}^{k_{r,g}^{(2)}} (\hat{s}_{r,g}^{-}(j), \hat{s}_{r,g}^{+}(j)), \quad (6)
\]
and
\[
\Omega_{r,g}^{0} = \bigcup_{j=1}^{m_{r,g}^{(1)}} (t_{r,g}^{-}(j), t_{r,g}^{+}(j)) \bigcup_{j=1}^{m_{r,g}^{(2)}} (\hat{t}_{r,g}^{-}(j), \hat{t}_{r,g}^{+}(j)). \quad (7)
\]

For the partition (5), the subintervals \((q_{r,g}^{-}(j), q_{r,g}^{+}(j)), j = 1, 2, \cdots, n_{r,g}^{(1)}\) and
\((\hat{q}_{r,g}^{-}(j), \hat{q}_{r,g}^{+}(j)), j = 1, 2, \cdots, n_{r,g}^{(2)}\) are countable mutually disjoint, and any two subintervals have no common edges. The above property also holds for the partitions (6) and (7).

**Remark 2.1.** Proposition 2.1 states that the interval \([0, 1]\) can be expressed as a union of mutually disjoint subintervals. In each subinterval, the sign of the function \(r - g\) keeps unchanged.

The following example shows that the number of the subintervals in (5) could be infinity.
Example 2.1. Assume that \( r(x) = (1 + \beta)x \) for \( \beta > 0 \). The function \( g(x) \) with \( g(0) = 1, g(1) = 1 \) is defined in the following way: for \( i = \lfloor \beta \rfloor + 1, \lfloor \beta \rfloor + 2, \ldots \), we let

\[
g\left(\frac{1}{2^n}\right) = \frac{1 + \beta}{2^{i+1}} - \frac{1 + \beta}{2^{i+1}}, \quad g\left(\frac{1}{2^{i+1}}\right) = \frac{1 + \beta}{2^{i+1}} + \frac{1 + \beta}{2^{i+1}},
\]

where \( \lfloor \beta \rfloor \) denotes the biggest integer less than or equal to \( \beta \), and the values at the other points in \((0, 1)\) are constructed by linear interpolation. Thus the function \( g(x) \) is increasing,

\[
\Omega_{r,g}^- = \{x: (1 + \beta)x - g(x) < 0\} = \bigcup_{i=\lfloor \beta \rfloor + 1}^{\infty} \left(\frac{18}{17} \cdot \frac{2^{i+2}}{2^{i+2}}, \frac{3}{2^{i+2}}\right)
\]

and

\[
\Omega_{r,g}^0 = \{0, \frac{18}{17} \cdot \frac{2^{i+2}}{2^{i+2}}, \frac{3}{2^{i+2}}, \ i = \lfloor \beta \rfloor + 1, \cdots \}.
\]

Next we will give some examples to explain the above partition. In the following two examples, the reinsurance premium principle is the expected premium principle with \( r(x) = (1 + \beta)x, \beta > 0 \) in (2), or Wang’s premium principle, that is, the function \( r(x), x \in [0, 1] \) is an increasing concave function such that \( r(0) = 0 \) and \( r(1) = 1 \). For a random variable \( X \) with distribution \( F_X \), its left-continuous inverse function \( Q_X(x) \) is defined as \( Q_X(x) = \inf\{y: F_X(y) \geq x\} \), and \( S_X(x) = P(X > x) \).

Example 2.2. VaR of the risk \( X \) with level \( \alpha \) is defined as

\[
VaR_{\alpha}(X) = Q_X(1 - \alpha).
\]

The corresponding distortion function can be represented as \( g(x) = I_{\{x>\alpha\}} \).

(1) Expected premium principle. In the case \( \alpha < \frac{1}{1+\beta} \),

\[
\Omega_{r,g}^- = (\alpha, \frac{1}{1+\beta}), \quad \Omega_{r,g}^0 = \{0, \frac{1}{1+\beta}\}.
\]

Otherwise,

\[
\Omega_{r,g}^- = \emptyset, \quad \Omega_{r,g}^0 = \{0\}.
\]
In practice, the parameter $\alpha$ is always much smaller than $\frac{1}{1+\beta}$.

(2) Wang’s premium principle. We have

$$
\Omega^-_{r,g} = (\alpha, 1), \quad \Omega^0_{r,g} = \{0, 1\}.
$$

**Example 2.3.** TVaR of a non-negative random variable $X$ at a confidence level $1 - \alpha, 0 < \alpha < 1$ is defined as

$$
TVaR_\alpha(X) = \frac{1}{\alpha} \int_{1-\alpha}^1 Q_X(x)dx.
$$

Note that $\rho_g[X] = TVaR_\alpha(X)$ with the concave function $g(x) = \min\{\frac{x}{\alpha}, 1\}$.

(1) Expected premium principle. If $\alpha < \frac{1}{1+\beta}$, then

$$
\Omega^-_{r,g} = (0, \frac{1}{1+\beta}), \quad \Omega^0_{r,g} = \{0, \frac{1}{1+\beta}\}.
$$

If $\alpha = \frac{1}{1+\beta}$, then

$$
\Omega^-_{r,g} = \emptyset, \quad \Omega^0_{r,g} = \left[0, \frac{1}{1+\beta}\right].
$$

If $\alpha > \frac{1}{1+\beta}$, then

$$
\Omega^-_{r,g} = \emptyset, \quad \Omega^0_{r,g} = \{0\}.
$$

(2) Wang’s premium principle. If $r'_+(0) > \frac{1}{\alpha}$, then there exists a unique $0 < z < \frac{1}{\alpha}$ such that $r(z) = \frac{z}{\alpha}$. And

$$
\Omega^-_{r,g} = (z, 1), \quad \Omega^0_{r,g} = \{0, z, 1\}.
$$

Otherwise, define $z_2 = \inf\{x \geq 0 : r(x) < g(x)\}$ then

$$
\Omega^-_{r,g} = (z_2, 1), \quad \Omega^0_{r,g} = [0, z_2] \cup \{1\}.
$$

**2.2 The optimal ceded loss function without premium constraint**

For the random variable $X \geq 0$, its survival function is defined as $S_X(x) = P(X > x)$. We define

$$
S_X^{-1}(a) = \inf\{x : S_X(x) \leq a\}, \quad S_X^{(-1)}(a) = \sup\{x : S_X(x) \geq a\}.
$$
Proposition 2.2. We have the following properties:

1. \( S_X^{-1}(a) \leq S_X^{(-1)}(a) \), \( S_X(S_X^{-1}(a)) \leq a \), and \( S_X(S_X^{(-1)}(a)) \leq a \);

2. \( S_X^{-1}(a) = S_X^{(-1)}(a) \) if and only if \( S_X(t) = a, t \geq 0 \) has at most one solution;

3. \( S_X(t) \leq a \) if and only if \( t \geq S_X^{-1}(a) \), and \( t > S_X^{(-1)}(a) \) implies \( S_X(t) < a \) while \( S_X(t) < a \) implies \( t \geq S_X^{(-1)}(a) \).

Write
\[
\Gamma_{r,g}^-(X) = \{ x : r(S_X(x)) - g(S_X(x)) < 0, \ x \geq 0 \},
\]
\[
\Gamma_{r,g}^0(X) = \{ x : r(S_X(x)) - g(S_X(x)) = 0, \ x \geq 0 \}
\]
and
\[
\Gamma_{r,g}^+(X) = \{ x : r(S_X(x)) - g(S_X(x)) > 0, \ x \geq 0 \}.
\]

By Proposition 2.2, we can derive \( S_X(t) \in (a,b) \iff t \in [S_X^{-1}(b), S_X^{-1}(a)] \) and \( t \in (S_X^{(-1)}(b), S_X^{(-1)}(a)) \Rightarrow S_X(t) \in (a,b) \Rightarrow t \in [S_X^{(-1)}(b), S_X^{(-1)}(a)] \). Together with the equation (5), we can obtain
\[
\Gamma_{r,g}^-(X) \supseteq \bigcup_{j=1}^{n_{r,g}} (S_X^{-1}(\hat{q}_{r,g}^+(j)), S_X^{-1}(\hat{q}_{r,g}^-(j))) \bigcup_{j=1}^{n_{r,g}} [S_X^{-1}(\hat{q}_{r,g}^+(j)), S_X^{-1}(\hat{q}_{r,g}^-(j))],
\]
meanwhile,
\[
\Gamma_{r,g}^-(X) \subseteq \bigcup_{j=1}^{n_{r,g}} (S_X^{-1}(q_{r,g}^+(j)), S_X^{-1}(q_{r,g}^-(j))) \bigcup_{j=1}^{n_{r,g}} [S_X^{-1}(q_{r,g}^+(j)), S_X^{-1}(q_{r,g}^-(j))].
\]

Therefore, there exists a set \( N_1 \) having null lebesgue measure such that the set \( \Gamma_{r,g}^-(X) \) can be rewritten as
\[
\Gamma_{r,g}^-(X) = \bigcup_{j=1}^{n_{r,g}} (S_X^{-1}(q_{r,g}^+(j)), S_X^{-1}(q_{r,g}^-(j))) \bigcup_{j=1}^{n_{r,g}} [S_X^{-1}(q_{r,g}^+(j)), S_X^{-1}(q_{r,g}^-(j))] \bigcup N_1.
\]

Similarly, there exist lebesgue null sets \( N_2 \) and \( N_3 \) such that
\[
\Gamma_{r,g}^0(X) = \bigcup_{j=1}^{n_{r,g}} (S_X^{-1}(t_{r,g}^+(j)), S_X^{-1}(t_{r,g}^-(j))) \bigcup_{j=1}^{n_{r,g}} [S_X^{-1}(t_{r,g}^+(j)), S_X^{-1}(t_{r,g}^-(j))] \bigcup N_2
\]
and
\[
\Gamma_{r,g}^\ast(X) = \bigcup_{j=1}^{n_{r,g}^{(1)}} (S_X^{-1}(s_{r,g}^+(j)), S_X^{-1}(s_{r,g}^-(j))) \cup \bigcup_{j=1}^{n_{r,g}^{(2)}} (S_X^{-1}(\hat{s}_{r,g}^+(j)), S_X^{-1}(\hat{s}_{r,g}^-(j))) \cup N_3.
\]

For simplicity, in the following discussion we define \( \inf 0 = \infty, \inf 0 = 0 \) and \( \min\{x - \infty, \infty - \infty\} = 0. \)

**Theorem 2.1.** We have
\[
\min_{f \in \mathcal{F}} \rho_g[T_f(X)] = \int_0^\infty \min\{r(S_X(t)), g(S_X(t))\} dt,
\]
and the optimum is attained at the ceded loss function
\[
f^\ast(x) = \sum_{i=1}^{n_{r,g}^{(1)}} \min\{[x - S_X^{-1}(q_{r,g}^+(i))], S_X^{-1}(q_{r,g}^-(i)) - S_X^{-1}(q_{r,g}^+(i))\}
+ \sum_{i=1}^{n_{r,g}^{(2)}} \min\{[x - S_X^{-1}(\hat{q}_{r,g}^+(i))], S_X^{-1}(\hat{q}_{r,g}^-(i)) - S_X^{-1}(\hat{q}_{r,g}^+(i))\}
+ \sum_{i=1}^{m_{r,g}^{(1)}} a_i \min\{[x - S_X^{-1}(t_{r,g}^+(i))], S_X^{-1}(t_{r,g}^+(i)) - S_X^{-1}(t_{r,g}^-(i))\}
+ \sum_{i=1}^{m_{r,g}^{(2)}} b_i \min\{[x - S_X^{-1}(\hat{t}_{r,g}^+(i))], S_X^{-1}(\hat{t}_{r,g}^+(i)) - S_X^{-1}(\hat{t}_{r,g}^-(i))\}. \tag{9}
\]

Here \( 0 \leq a_i, b_i \leq 1 \) are arbitrary constants, the points \( \{q_{r,g}^+(i), q_{r,g}^-(i); i = 1, \ldots, n_{r,g}^{(1)}\} \) and \( \{\hat{q}_{r,g}^+(i), \hat{q}_{r,g}^-(i); i = 1, \ldots, n_{r,g}^{(2)}\} \) are given by equation (5), and \( \{t_{r,g}^+(i), t_{r,g}^-(i); i = 1, \ldots, m_{r,g}^{(1)}\} \) and \( \{\hat{t}_{r,g}^+(i), \hat{t}_{r,g}^-(i); i = 1, \ldots, m_{r,g}^{(2)}\} \) are given by equation (7).

By comparing the functions \( r(S_X(x)), x \geq 0 \) and \( g(S_X(x)), x \geq 0 \), the optimal ceded loss function \( f^\ast(x) \) can be explained as the followings:

- Case one: \( \{x : r(S_X(x)) - g(S_X(x)) < 0, x \geq 0\} \). In the set the value of the risk function \( g(S_X(\cdot)) \) is bigger than the premium function \( r(S_X(\cdot)) \). Thus the optimal choice for the insurer is to cede all the increasing part of losses in these layers. The first two terms of the righthand of (9) correspond to the set.
• Case two: \( \{ x : r(S_X(x)) - g(S_X(x)) = 0, \ x \geq 0 \} \). In the set the value of the risk function \( g(S_X(\cdot)) \) and the premium function \( r(S_X(\cdot)) \) are equal. Thus it makes no difference to cede how much the proportion of such layer to the reinsurer. The last two terms of the righthand of (9) correspond to the set.

• Case three: \( \{ x : r(S_X(x)) - g(S_X(x)) > 0, \ x \geq 0 \} \). In the set the value of the risk function \( g(S_X(\cdot)) \) is smaller than the premium function \( r(S_X(\cdot)) \). Thus the optimal choice for the insurer is to retain all the increasing part of losses in these layers. No terms in (9) correspond to the set.

The interaction between the reinsurance premium principle (2) and the risk measure (1) makes the optimal reinsurance treaty a little complicated. By balancing the reinsurance premium payment and the insurer’s risk, the optimal reinsurance arrangement is divided into some layers. In each layer, the optimal reinsurance strategy can be either limited stop-loss or a combination of quota share with limited stop-loss. Moreover, the last two terms of the righthand of (9) do not increase the value of the insurer’s risk measure, but it increases the reinsurance premium. In the case \( a_i = 0, b_i = 0, i \geq 1 \), the reinsurer’s premium is the smallest among all \( f^*(x) \)’s.

In order to make the above definitions clearly, we give Figure 2.1 to show the relationship between \( \Omega_{r,g}^- \) and the optimal function \( f^*(x) \). Note that for the functions \( g \) and \( r \) in the figure, \( q^{-}_{r,g}(1) = 0 \),

\[
\Omega_{r,g}^- = \bigcup_{i=1}^{3} (q^{-}_{r,g}(i), q^{+}_{r,g}(i)), \quad \Omega_{r,g}^0 = \{ q^{-}_{r,g}(i), q^{+}_{r,g}(j), i = 1, 2, j = 1, 2, 3 \},
\]

and

\[
f^*(x) = \sum_{i=1}^{3} \min\{ [x - S^{-1}_X(q^{+}_{r,g}(i))]_+, S^{-1}_X(q^{-}_{r,g}(i)) - S^{-1}_X(q^{+}_{r,g}(i)) \}.
\]

Remark 2.2. (1) For the optimal ceded loss function \( f^*(x) \) in (9), the two inverse functions \( S^{-1}_X(\cdot) \) and \( S^{-1}_X(\cdot) \) were used. Note that in some cases the two functions are equal. See Proposition 2.2.
(2) For two premium principle functions $r_1$ and $r_2$ with $r_1(x) \leq r_2(x), x \geq 0$, from (8) we know that under the same risk measure $\rho_g$ we have

$$\min_{f \in F} \rho_g [X - f(X) + \mu_{r_1}(f(X))] \leq \min_{f \in F} \rho_g [X - f(X) + \mu_{r_2}(f(X))].$$

It means that for the smaller $r$, the corresponding optimal risk measure is smaller also.

Similarly, for the two distortion functions $g_1(x) \leq g_2(x), x \in [0, 1]$ and the premium principle function $r$,

$$\min_{f \in F} \rho_{g_1} [X - f(X) + \mu_{r}(f(X))] \leq \min_{f \in F} \rho_{g_2} [X - f(X) + \mu_{r}(f(X))].$$

### 2.3 The optimal reinsurance treaty with premium constraint

In the following, we consider the optimal problem (4). For the optimal function $f^*(X)$ given in Theorem 2.1, when $\mu_r(f^*(X)) \leq \pi$ the ceded function $f^*(x)$ is one optimal solution of the optimization problem (4). In the next we consider the case that $\mu_r(f^*(X)) > \pi$.

First, we need to find one $\theta > 0$ according to the reinsurance premium constraint $\mu_r(f(X)) \leq \pi$. 

Figure 2.1: The functions $r, g$ and the corresponding optimal ceded loss function $f^*$
Lemma 2.1. Suppose that $\mu_r(f^*(X)) > \pi$. Define

$$\theta = \inf\{a > 0 : \int_{(1+a)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t))dt \leq \pi\}.$$ 

- If there exists $a > 0$ such that $\int_{(1+a)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t))dt = \pi$, then

$$\int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t))dt = \pi.$$ (10)

- If the equation $\int_{(1+a)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t))dt = \pi$ has no solution, then

$$\int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t))dt < \pi \leq \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) \leq 0} r(S_X(t))dt.$$ (11)

For the given $\theta \geq 0$, denote

$$\pi_1 = \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t))dt, \quad \pi_2 = \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) \leq 0} r(S_X(t))dt.$$ 

When (10) holds, we have $\pi_1 = \pi = \pi_2$. Otherwise, $\pi_1 < \pi \leq \pi_2$ follows.

Similarly as $\Gamma_{r,g}^{-}(X)$ and $\Gamma_{r,g}^{0}(X)$, we can write

$$\Gamma_{(1+\theta)r,g}^{-}(X) = \{x : (1 + \theta)r(S_X(x)) - g(S_X(x)) < 0, \ x \geq 0\}$$ 

and

$$\Gamma_{(1+\theta)r,g}^{0}(X) = \{x : (1 + \theta)r(S_X(x)) - g(S_X(x)) = 0, \ x \geq 0\}.$$ 

And we can get the following partitions

$$\Gamma_{(1+\theta)r,g}^{-}(X) = \bigcup_{j=1}^{n_{1}(1+\theta)r,g} (S_X^{-1}((q_{(1+\theta)r,g}^{+}(j)), S_X^{-1}((q_{(1+\theta)r,g}^{-}(j))))$$

$$\bigcup_{j=1}^{n_{2}(1+\theta)r,g} (S_X^{-1}((q_{(1+\theta)r,g}^{+}(j)), S_X^{-1}((q_{(1+\theta)r,g}^{-}(j)))) \bigcup N_4$$ (12)

and

$$\Gamma_{(1+\theta)r,g}^{0}(X) = \bigcup_{j=1}^{m_{1}(1+\theta)r,g} [S_X^{-1}((t_{(1+\theta)r,g}^{+}(j)), S_X^{-1}((t_{(1+\theta)r,g}^{-}(j))))$$

$$\bigcup_{j=1}^{m_{2}(1+\theta)r,g} [S_X^{-1}((t_{(1+\theta)r,g}^{+}(j)), S_X^{-1}((t_{(1+\theta)r,g}^{-}(j)))) \bigcup N_5,$$ (13)
where $N_4$ and $N_5$ are lebesgue null sets.

Based on the above partitions, the function $f^{**}(x) = f_1^{**}(x) + f_2^{**}(x)$ with $f_1^{**}(x) + f_2^{**}(x) \in \mathcal{F}$ and $\mu_r(f_1^{**}(x) + f_2^{**}(x)) = \pi$ can be defined. The function $f_1^{**}(x)$ is defined according to the partition $\Gamma_{(1+\theta)r,g}^0(X)$,

$$f_1^{**}(x) = \sum_{i=1}^{n_{(1+\theta)r,g}^{(1)}} \min\{(x - S_X^{(-1)}(q_{(1+\theta)r,g}^+(i))) \}, (S_X^{(-1)}(q_{(1+\theta)r,g}^-(i)) - S_X^{(-1)}(q_{(1+\theta)r,g}^+(i)))\}$$

Similarly, the function $f_2^{**}(x) \geq 0$ is defined according to the partition $\Gamma_{(1+\theta)r,g}^0(X)$,

$$f_2^{**}(x) = \sum_{i=1}^{n_{(1+\theta)r,g}^{(1)}} \min\{(x - S_X^{(-1)}(q_{(1+\theta)r,g}^+(i))) \}, (S_X^{(-1)}(q_{(1+\theta)r,g}^-(i)) - S_X^{(-1)}(q_{(1+\theta)r,g}^+(i)))\}$$

The following theorem states that $f^{**}(x)$ is one optimal solution for the optimal problem (4) when $\mu_r(f^*(X)) > \pi$.

**Theorem 2.2.** (1) In the case $\mu_r(f^*(X)) > \pi$ for $f^*$ in Theorem 2.1, the function $f^{**}(x)$ is one solution of the optimization problem (4). The minimal risk measure can be expressed as

$$\rho_g[T_{f^{**}}(X)] = \rho_g[X] + \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} [r(S_X(t)) - g(S_X(t))] dt - \theta(\pi - \pi_1).$$

(2) In the case $\mu_r(f^*(X)) \leq \pi$ for $f^*$ in Theorem 2.1, $f^*$ is one solution of the optimization problem (4).

**Remark 2.3.** The minimal distortion risk measure with premium restriction can be rep-
resented as

\[\rho_{g}[T_{f^{**}}(X)] = \int_{(1+\theta)r(S_X(t))-g(S_X(t))<0} r(S_X(t)) dt + \int_{(1+\theta)r(S_X(t))-g(S_X(t))>0} g(S_X(t)) dt + \frac{\pi - \pi_1}{\pi_2 - \pi_1} \int_{(1+\theta)r(S_X(t))=0} r(S_X(t)) dt + \frac{\pi_2 - \pi}{\pi_2 - \pi_1} \int_{(1+\theta)r(S_X(t))=0} g(S_X(t)) dt. \]  

(15)

In the presence of reinsurance premium restriction and \( \mu_{r}(f^{*}(X)) > \pi \), ceding all the excess losses on the layers where \( \{x : (1 + \theta)r(S_X(x)) - g(S_X(x)) < 0, \ x \geq 0\} \) will be infeasible. Thus we need to increase the threshold of reinsurance to satisfy the reinsurance premium restriction through finding one \( \theta \geq 0 \). By using the constant \( \theta \), we can divide the range of initial loss into three mutually disjoint sets, \( \{x : (1 + \theta)r(S_X(x)) - g(S_X(x)) < 0, \ x \geq 0\} \), \( \{x : (1 + \theta)r(S_X(x)) - g(S_X(x)) = 0, \ x \geq 0\} \) and \( \{x : (1 + \theta)r(S_X(x)) - g(S_X(x)) > 0, \ x \geq 0\} \).

• Case one: \( \{x : (1 + \theta)r(S_X(x)) - g(S_X(x)) < 0, \ x \geq 0\} \). In such layers, it is obvious that \( r(S_X(x)) - g(S_X(x)) < 0 \). Thus, from the discussion in Theorem 2.1, ceding the whole excess losses in these layers to the reinsurer will reduce the insurer’s risk most. The term \( f^{**}_{1}(x) \) corresponds to this ceding. As a result, on these excess-of-loss layers, reinsurance premium function \( r(S_X(\cdot)) \) is used to measure the insurer’s total payment to the reinsurer. The first term of the righthand of (15) measures the insurer’s risk in this case;

• Case two: \( \{x : (1 + \theta)r(S_X(x)) - g(S_X(x)) = 0, \ x \geq 0\} \). In such layers, for \( x > 0 \) there will always be \( r(S_X(x)) - g(S_X(x)) < 0 \). When ignoring reinsurance premium restriction, ceding all the excess losses to the reinsurer will be a optimal choice, but the final reinsurance premium might exceed the reinsurance constraint. In order to satisfy the reinsurance premium restriction, the insurer should cede some part of the excess loss on each layer to the reinsurer until the paid reinsurance
premium equals the restriction level. The term $f^{**}_{2}(x)$ corresponds to this ceding. Note that the coefficient $\frac{\pi - \pi_1}{\pi_2 - \pi_1}$ in $f^*_2(x)$ reflects the proportion of loss ceded to the reinsurer. Therefore, on the ceded parts of these excess of loss layers, the reinsurance premium function $r(S_X(\cdot))$ is used for measuring the insurer’s total payment, the third term of the righthand of (15) corresponds to this part. For the retained parts of loss in these layers, the insurer’s risk is measured by risk function $g(S_X(\cdot))$ and the last term of the righthand of (15) corresponds to this part;

- Case three: \{\(x : (1 + \theta)r(S_X(x)) - g(S_X(x)) > 0, \ x \geq 0\}\}. As stated before, the premium charged for the ceded losses in case one and case two has already achieved the reinsurance premium’s restriction level. Thus the insurer needs to retain all the excess losses in these layers, and the function $g(S_X(\cdot))$ is used to measure the insurer’s total payment in these layers, and the second term of the righthand of (15) corresponds to this case.

Meanwhile, when $\pi = \pi_1$, there is $f^{**}_{2}(x) = 0$ and we need no adjustment in \{\(x : (1 + \theta)r(S_X(x)) - g(S_X(x)) = 0, \ x \geq 0\}\} to reflect the reinsurance premium’s constraint.

An alternative choice of $f^{**}$ can be constructed as follows.

**Theorem 2.3.** Let $\bigcup_{i=1}^{\kappa}(u^-{(i)}, u^+(i))$ be a union of disjoint subintervals,

$$\Gamma_{(1+\theta)r,g}(X) \subseteq \bigcup_{i=1}^{\kappa}[u^-(i), u^+(i)]$$

and

$$\bigcup_{i=1}^{\kappa}(u^-{(i)}, u^+(i)) \subseteq \Gamma_{(1+\theta)r,g}(X) \bigcup \Gamma_{0(1+\theta)r,g}(X).$$

When $\mu_r(f^{*}(X)) > \pi$, the ceded loss function

$$f^{**}(x) = \sum_{i=1}^{\kappa} \min\{x - u^-(i), u^+(i) - u^-(i)\}$$

with $\mu_r(f^{**}(X)) = \pi$ is one solution of the optimization problem (4).
As we mentioned before, the former papers suggested that the optimal reinsurance strategy could be one of the main reinsurance strategies: quota share, stop-loss, limited stop-loss, or their hybrids. Note that many insurance companies, especially those whose insurance coverage conclude catastrophe risks, have more complicated reinsurance strategy. These insurance companies always partition the ranges of their claims into several layers. For the different layers the insurer may adopt distinct reinsurance strategies. Froot (2001) provided a simple description of USAA’s 1997 reinsurance program. There are several layers in the program, at each layer USAA cedes a portion of layer to a reinsurance company. Theorem 2.1 and Theorem 2.2 provide some insight about the program. The essence of the theorems is that the range of insurer’s loss can be divided into several layers, and the insurer will face a tradeoff on each layer between ceded loss and retained loss.

3 Expected premium principle and Wang’s premium principle

Due to the desirable properties of expected premium principle and Wang’s premium principle, many papers on optimal reinsurance focused on the two premium principles. As the two special cases of (2), some existing results on the two premium principles can also be derived directly from our results. This section will introduce the current main results on the two premium principles briefly, and analysis the influence of the premium principle on the optimal reinsurance strategy.

3.1 Brief introduction on the current development

Define

\[ \mathcal{H} = \{0 \leq f(x) \leq x : f(x) \text{ is an increasing convex function}\}. \]
The set \( \mathcal{H} \) is a subset of \( \mathcal{F} \), see Lemma A.1 in Cai et al. (2008) for details. The convexity assumption on ceded loss functions would exclude a large family of reinsurance strategies, such as limited stop-loss reinsurance. Note that the limited stop-loss functions belong to the class \( \mathcal{F} \).

As introduced before, some recent developments on VaR and TVaR optimization problems under the expected premium principle include Cai et al. (2008), Cheung (2010), Chi and Tan (2010), Tan, Weng and Zhang (2011) and some references therein. Cai et al. (2008) and Cheung (2010) concerned VaR and TVaR optimization problems when the reinsurance premium was calculated by expected premium principle, and they derived the optimal reinsurance designs in the class \( \mathcal{H} \). Besides expected premium principle, Cheung (2010) also concerned Wang’s premium principle and found the optimal solution to the same VaR optimization problems in \( \mathcal{H} \). Tan, Weng and Zhang (2011) studied TVaR optimization problems over the ceded loss function’s family \( \{ f(x), x \geq 0 : 0 \leq f(x) \leq x \} \) and proved that the stop-loss reinsurance is optimal. The TVaR optimization problem was also solved by Chi and Tan (2010) using stochastic order and they derived the solution of VaR optimization problem in different families including \( \mathcal{H} \) and \( \mathcal{F} \) as well. Chi and Tan (2011) concerned the VaR and TVaR optimization problems for a class of premium principles. Based on Theorems 2.1-2.3, some former results in the family \( \mathcal{F} \) can be derived.

In the next, we will concentrate on two distortion risk measures: VaR and concave distortion risk measures. The word "concave" in concave distortion risk measure means that the distortion function is concave. It is known that TVaR, Wang’s transform risk measure and Beta distortion risk measure belong to the family of concave distortion measures. The reinsurance premium is calculated by the expected premium principle or Wang’s premium principle. The optimal problems we consider in this section can be presented as follows.
**VaR optimization:**

\[
\min_{f \in \mathcal{F}} \{ VaR_\alpha[T_f(X)] \} \tag{16}
\]

and

\[
\begin{cases}
\min_{f \in \mathcal{F}} \{ VaR_\alpha[T_f(X)] \} \\
\text{s.t.} \quad \mu_r(f(X)) \leq \pi.
\end{cases} \tag{17}
\]

**Concave distortion risk measure optimization:**

\[
\min_{f \in \mathcal{F}} \{ \rho[g[T_f(X)]] \} \tag{18}
\]

and

\[
\begin{cases}
\min_{f \in \mathcal{F}} \{ \rho[g[T_f(X)]] \} \\
\text{s.t.} \quad \mu_r(f(X)) \leq \pi,
\end{cases} \tag{19}
\]

where the associated distortion function \( g(x) \) is concave.

When we apply the expected premium principle to calculate reinsurance premium, the reinsurance premium restriction can be represented as \( E[f(X)] \leq B \), where \( B = \pi/(1 + \beta) \).

### 3.2 VaR-optimization

By Theorems 2.1-2.3, we can derive the following corollary for the VaR optimization problems (16) and (17).

**Corollary 3.1.** 1. Consider the expected premium principle.

(a) For VaR-optimization problem (16),

\[
f^*_{E, VaR}(x) = \min \{ (x - S_X^{-1}(1/(1+\beta)))_+, S_X^{-1}(\alpha) - S_X^{-1}(1/(1+\beta)) \}
\]

is one optimal solution;
(b) For VaR-optimization problem (17), if $E[f^{E,VaR}_{E}(X)] \leq B$, then $f^{E,VaR}_{E}(x)$ is one optimal solution; otherwise, the following two ceded loss functions $f^{**}_{E,VaR}(x)$ and $f^{***}_{E,VaR}(x)$ are two optimal solutions,

$$f^{**}_{E,VaR}(x) = \min\{(x - S^{-1}_X(\frac{1}{1+\beta}))_+, S^{-1}_X(\frac{1}{1+\beta}) - S^{-1}_X(\frac{1}{1+\beta})\} + \lambda_1 \min\{(x - S^{-1}_X(\frac{1}{1+\beta}))_+, S^{-1}_X(\frac{1}{1+\beta}) - S^{-1}_X(\frac{1}{1+\beta})\},$$

where $\tilde{\beta}$ is determined by $\int_{S_X^{-1}(\frac{1}{1+\beta})}^{S_X(\alpha)} S_X(t) dt \leq B \leq \int_{S_X^{-1}(\frac{1}{1+\beta})}^{S_X(\alpha)} S_X(t) dt$ and $0 \leq \lambda_1 \leq 1$ is determined by $E[f^{**}_{E,VaR}(X)] = B$, and

$$f^{***}_{E,VaR}(x) = \min\{(x - d^*_1)_+, S^{-1}_X(\alpha) - d^*_1\},$$

where $d^*_1$ is determined by $E[f^{***}_{E,VaR}(X)] = B$.

2. Consider Wang’s premium principle.

(a) For VaR optimization problem (16),

$$f^{W,VaR}_{W}(x) = x \wedge S^{-1}_X(\alpha)$$

is one optimal solution;

(b) For VaR-optimization problem (17), if $\mu_r(f^{W,VaR}_{W}(X)) \leq \pi$, then $f^{W,VaR}_{W}(x)$ is one optimal solution. Otherwise,

$$f^{**}_{W,VaR}(x) = \min\{(x - d^*_2)_+, S^{-1}_X(\alpha) - d^*_2\}$$

is one optimal solution, where $d^*_2$ is determined by $\mu_r(f^{**}_{W,VaR}(X)) = \pi$.

From the above corollary we know that the insurer’s VaR attains the same minimum at two different solutions $f^{**}_{E,VaR}(x)$ and $f^{***}_{E,VaR}(x)$. Note that the constant $d^*_1 \in [S^{-1}_X(\frac{1}{1+\beta}), S^{-1}_X(\frac{1}{1+\beta})]$, implying that the two optimal loss functions $f^{**}_{E,VaR}(x)$ and $f^{***}_{E,VaR}(x)$ are the same when $S^{-1}_X(\frac{1}{1+\beta}) = S^{-1}_X(\frac{1}{1+\beta})$.
Remark 3.1. With the reinsurance premium calculated by the expected premium principle, Chi and Tan (2010) considered the VaR optimization problem under the premium constraint $E[f(X)] = B$. In contrast, we use the condition $E[f(X)] \leq B$ instead. In some cases the two different assumptions can lead to different optimal ceded loss functions.

Remark 3.2. Comparing with the expected premium principle, Wang’s premium principle charges less for small loss. When there is no premium restriction, the insurer cedes all the small loss will be a wise choice. At the same time, when the premium principle is expected premium principle, the insurer may cover small losses itself.

3.3 Concave distortion risk measure

For the concave distortion risk measure optimization problems (18) and (19), the following results can be derived from Theorem 2.1 and Theorem 2.3.

Corollary 3.2. Consider the expected premium principle.

1. Define $q^*_3 = \inf\{x : (1 + \beta)x \geq g(x)\}$, then

$$f^*_{E,\text{Conc}}(x) = (x - S^{-1}_X(q^*_3))_+$$

is one solution for the concave distortion risk measure optimization problem (18).

2. For the concave distortion risk measure optimization problem (19), if $E[f^*_{E,\text{Conc}}(X)] \leq B$, then $f^*_{E,\text{Conc}}(x)$ is one optimal solution; otherwise,

$$f^{**}_{E,\text{Conc}}(x) = (x - d^*_3)_+$$

is one optimal solution, where $d^*_3$ is determined by $E[f^{**}_{E,\text{Conc}}(X)] = B$.

For Wang’s premium principle, the solutions of optimization problems (18) and (19) have the same forms as in Theorem 2.1 and Theorem 2.3, thus we will not give its optimal reinsurance here.
Remark 3.3. The above corollary states that, when the reinsurance premium is calculated by the expected premium principle, stop-loss reinsurance is always optimal in the sense of minimizing the concave distortion risk measure. As we mentioned before, TVaR, Wang’s transform risk measure and Beta distortion risk measure belong to the class of concave distortion risk measures. Thus for a large family of widely used risk measures, stop-loss reinsurance is optimal.

Since TVaR is one of widely used concave distortion risk measures, we will give the optimal solution of TVaR optimization problem in the following example. For simplicity, we only consider the TVaR optimal problem with premium constraint.

Example 3.1. (TVaR optimization problem). For the TVaR of a confidence level $1 - \alpha$ where $0 < \alpha < 1$, it is known that $g(x) = \min\{\frac{x}{\alpha}, 1\}$.

1. Consider the expected premium principle. When $\alpha < \frac{1}{1+\beta}$, we have $\inf\{x : (1 + \beta)x \geq g(x)\} = \frac{1}{1+\beta}$. And $\inf\{x : (1 + \beta)x \geq g(x)\} = 0$ if $\alpha \geq \frac{1}{1+\beta}$. The optimal reinsurance treaty is listed in the following table:

<table>
<thead>
<tr>
<th>Conditions</th>
<th>The optimal ceded loss function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1: $\alpha &lt; \frac{1}{1+\beta}$, $\int_{S_X^{-1}(\frac{1}{1+\beta})}^{\infty} S_X(t)dt \leq B$</td>
<td>$(x - S_X^{-1}(\frac{1}{1+\beta}))_+$</td>
</tr>
<tr>
<td>Case 2: $\alpha &lt; \frac{1}{1+\beta}$, $\int_{S_X^{-1}(\frac{1}{1+\beta})}^{\infty} S_X(t)dt &gt; B$</td>
<td>$(x - d_4^*)_+$</td>
</tr>
<tr>
<td>Case 3: $\alpha \geq \frac{1}{1+\beta}$</td>
<td>0</td>
</tr>
</tbody>
</table>

where $d_4^*$ is determined by $\int_{d_4^*}^{\infty} S_X(t)dt = B$.

2. Consider Wang’s premium principle. Define $q_{5}^* = \inf\{x : r(x) < g(x)\}$. The optimal reinsurance treaty can be presented in the following table:
Table 3.2: Wang’s premium principle

<table>
<thead>
<tr>
<th>Conditions</th>
<th>The optimal ceded loss function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>$\int_0^{S_X^{-1}(q_0^*)} r(S_X(t))dt \leq \pi$</td>
</tr>
<tr>
<td>Case 2</td>
<td>$\int_0^{S_X^{-1}(q_0^*)} r(S_X(t))dt &gt; \pi$</td>
</tr>
</tbody>
</table>

When $\int_0^{S_X^{-1}(q_0^*)} r(S_X(t))dt > \pi$, we determine one $\theta > 0$ by Lemma 2.1. From Theorem 2.3, we can find $d_6^t$ and $d_7^t$ by $\{t : (1 + \theta)r(S_X(t)) < g(S_X(t))\} \subseteq [d_6^t, d_7^t]$, $(d_6^t, d_7^t) \subseteq \{t : (1 + \theta)r(S_X(t)) \leq g(S_X(t))\}$ and $\int_{d_6^t}^{d_7^t} r(S_X(t))dt = \pi$.

For the TVaR optimization, since Wang’s premium principle focuses more attention on large loss, then the insurer needs to pay more to the reinsurer for larger loss. Therefore, to minimize the insurer’s risk, a stop-loss with an upper limit will be a better choice. For the expected premium principle, the optimal reinsurance treaty always has a stop-loss form.

4 Proofs of main results

In this section we give the proofs of our results.

4.1 Proofs of Propositions 2.1-2.2

Proof of Proposition 2.1. Consider a fixed $0 < x_0 < 1$ such that $r(x_0) - g(x_0) < 0$. Since $r(x)$ and $g(x)$ are left continuous, there exists a $\delta > 0$ such that for all $y \in (x_0 - \delta, x_0]$, $r(y) - g(y) < 0$. We define

$$q_{r,g}^-(x_0) = \inf\{a : x_0 \in (a, b] \text{ and } r(y) - g(y) < 0, y \in (a, b)\}$$

and

$$q_{r,g}^+(x_0) = \sup\{b : x_0 \in (a, b] \text{ and } r(y) - g(y) < 0, y \in (a, b)\}.$$
Since \( r(x) \) and \( g(x) \) are left-continuous, we have

\[
\Omega_{r,g}^{-} = \bigcup_{x \in \Omega^{-}(r,g); r(q_{r,g}^{+}(x)) - g(q_{r,g}^{-}(x)) < 0} (q_{r,g}^{-}(x), q_{r,g}^{+}(x)) \bigcup_{x \in \Omega^{-}(r,g); r(q_{r,g}^{+}(x)) - g(q_{r,g}^{-}(x)) \geq 0} (q_{r,g}^{-}(x), q_{r,g}^{+}(x)).
\]

Then we can find countable mutually disjoint sets \((q_{r,g}^{-}(x_j), q_{r,g}^{+}(x_j)), j = 1, 2, \ldots, n_{r,g}^{(1)}\) and \((\hat{q}_{r,g}^{-}(x_j), \hat{q}_{r,g}^{+}(x_j)), j = 1, 2, \ldots, n_{r,g}^{(2)}\) such that

\[
\Omega_{r,g}^{-} = \bigcup_{j=1}^{n_{r,g}^{(1)}} (q_{r,g}^{-}(x_j), q_{r,g}^{+}(x_j)) \bigcup_{j=1}^{n_{r,g}^{(2)}} (\hat{q}_{r,g}^{-}(x_j), \hat{q}_{r,g}^{+}(x_j)),
\]

here \(n_{r,g}^{(1)}\) or \(n_{r,g}^{(2)}\) can be infinity. One can see that any two sets have no common edges.

For simplicity, we denote \((q_{r,g}^{-}(x_j), q_{r,g}^{+}(x_j))\) as \((q_{r,g}^{-}(j), q_{r,g}^{+}(j))\) and \((\hat{q}_{r,g}^{-}(x_j), \hat{q}_{r,g}^{+}(x_j))\) as \((\hat{q}_{r,g}^{-}(j), \hat{q}_{r,g}^{+}(j))\). Then (5) follows from the above equation.

Similarly, we can prove (6) and (7). \(\Box\)

**Proof of Proposition 2.2.**  (1) For any \( x < S_{X}^{-1}(a) \), there must be \( S_{X}(x) > a \).

Since \( S_{X}(x) \) is right continuous, there exists a \( \varepsilon > 0 \) such that \( S_{X}(x + \varepsilon) > a \), which means \( x + \varepsilon \in \{ y : S_{X}(y) \geq a \} \). Thus \( x + \varepsilon \leq S_{X}^{(-1)}(a) \) and therefore \( x < S_{X}^{(-1)}(a) \) follows. Now we conclude that \( S_{X}^{-1}(a) \leq S_{X}^{(-1)}(a) \).

Next, we first prove \( S_{X}(S_{X}^{-1}(a)) \leq a \). From the definition of \( S_{X}^{(-1)}(\cdot) \), we know that there exists sequence \( x_n, n \geq 1 \) such that \( x_n \downarrow S_{X}^{-1}(a) \) and \( S_{X}(x_n) \leq a \). From the right-continuity of \( S_{X}(x) \), we have \( S_{X}(S_{X}^{-1}(a)) = \lim_{x_n \downarrow S_{X}^{-1}(a)} S_{X}(x_n) \leq a \). Meanwhile, \( S_{X}(S_{X}^{-1}(a)) \leq a \) is obtained from the fact that \( S_{X}^{-1}(a) \leq S_{X}^{(-1)}(a) \) and \( S_{X}(x) \) is a decreasing function.

(2) We will prove that \( S_{X}^{(-1)}(a) < S_{X}^{(-1)}(a) \) if and only if \( S_{X}(t) = a \) have more than one solution.

(a) If \( S_{X}^{-1}(a) < S_{X}^{(-1)}(a) \), there exist some \( x_1 \) and \( x_2 \) such that \( S_{X}^{-1}(a) < x_1 < x_2 < S_{X}^{(-1)}(a) \). Since \( x_1 > S_{X}^{-1}(a) \Rightarrow S_{X}(x_1) \leq a \) and \( x_1 < S_{X}^{(-1)}(a) \Rightarrow S_{X}(x_1) \geq a \), we obtain \( S_{X}(x_i) = a \) for \( i = 1, 2 \), which means \( S_{X}(t) = a \) have more than one solution.
(b) If \( S_X(t) = a, t \geq 0 \) has more than one solution, without loss of generality, suppose there exist \( x_1 < x_2 \) and \( S_X(x_1) = S_X(x_2) = a \). From the definition of \( S_X^{-1}(a) \) and \( S_X^{(-1)}(a) \), there are \( x_1 \geq S_X^{-1}(a) \) and \( x_2 \leq S_X^{(-1)}(a) \). Thus \( S_X^{-1}(a) < S_X^{(-1)}(a) \).

(3) Note that \( S_X(x) \leq a \Rightarrow x \geq S_X^{-1}(a) \) is obvious. If \( S_X(x) > a \), by the right continuity of \( S_X(x) \), there exists a \( \varepsilon > 0 \) such that \( S_X(x + \varepsilon) > a \), then \( x + \varepsilon \leq S_X^{-1}(a) \) and therefore \( x < S_X^{-1}(a) \). Thus \( S_X(x) \leq a \Leftrightarrow x \geq S_X^{-1}(a) \) is proved.

If \( x > S_X^{(-1)}(a) \), then \( x \in \{ y : S_X(y) \geq a \} \) and therefore \( S_X(x) < a \). If \( S_X(x) < a \) then \( x \notin \{ y : S_X(y) \geq a \} \) and therefore \( x \geq S_X^{(-1)}(a) \). Now we get that

\[
x > S_X^{(-1)}(a) \Rightarrow S_X(x) < a \Rightarrow x \geq S_X^{(-1)}(a).
\]

\[ \square \]

4.2 Proof of Theorems 2.1-2.3

In order to prove Theorems 2.1-2.3, we need one lemma.

**Lemma 4.1.** For increasing continuous function \( f \), we have

\[
\rho_g[T_f(X)] = \rho_g[X] + \int_{\mathbb{R}^+} G_X(t)df(t),
\]  

(22)

where \( G_X(t) = r(S_X(t)) - g(S_X(t)) \).

**Proof.** Two random variables \( U \) and \( V \) are called comonotonic if there exist increasing functions \( f, l \) and a random variable \( Z \) such that \( U = f(Z), V = l(Z) \). For the distortion measure, Dhaene et al. (2006) proved that when \( U \) and \( V \) are comonotonic, then

\[
\rho_g[U + V] = \rho_g[U] + \rho_g[V].
\]  

(23)

Since \( f(x) \) and \( I_f(x) \) are increasing functions, thus \( f(X) \) and \( I_f(X) \) are comonotonic and

\[
\rho_g[X] = \rho_g[f(X)] + \rho_g[I_f(X)].
\]

26
Similarly,

\[ \rho_g[T_f(X)] = \mu_r(f(X)) + \rho_g[I_f(X)]. \]

Combining the above two equations we get

\[ \rho_g[T_f(X)] = \rho_g[X] + \mu_r(f(X)) - \rho_g[f(X)]. \] (24)

Write \( f^{-1}(x) = \inf\{t : f(t) > x\} \). It is easy to show that

\[ \{\omega : f(X(\omega)) > t\} = \{\omega : X(\omega) > f^{-1}(t)\}. \]

Therefore, using (24) we have

\[ \rho_g[T_f(X)] = \rho_g[X] - \int_0^\infty g(P(f(X) > t))dt + \int_0^\infty r(P(f(X) > t))dt \]
\[ = \rho_g[X] + \int_0^\infty r(P(X > f^{-1}(t)))dt - \int_0^\infty g(P(X > f^{-1}(t)))dt \]
\[ = \rho_g[X] + \int_0^\infty \{r(S_X(f^{-1}(t))) - g(S_X(f^{-1}(t)))\}dt \]
\[ = \rho_g[X] + \int_0^\infty G_X(f^{-1}(t))dt = \rho_g[X] + \int_{\mathbb{R}^+} G_X \circ f^{-1} d\mu, \] (25)

where \( \mu \) is the lebesgue measure. From the definition of \( f^{-1} \), we know it is a measurable map from \((\bar{\mathbb{R}}^+, \mathcal{B}_{\bar{\mathbb{R}}^+})\) to \((\mathbb{R}^+, \mathcal{B}_{\mathbb{R}^+})\). Using Theorem 6 in Chapter 4 of Rao (1987), there exists a measure \( \nu \) on \( \mathcal{B}_{\mathbb{R}^+} \) such that \( \mu(B) = \mu\{x : f^{-1}(x) \in B\} \) and the equation

\[ \int_{\mathbb{R}^+} G_X \circ f^{-1} d\mu = \int_{\mathbb{R}^+} G_X d\nu \] (26)

holds. For any \([a, b) \in \mathcal{B}_{\mathbb{R}^+} \), since \( f(s) > t \iff s > f^{-1}(t) \), we have \( \nu([a, b)) = \mu\{x : a \leq f^{-1}(x) < b\} = \mu\{x : f(a) \leq x < f(b)\} = f(b) - f(a) \). Note that \( f(x) \) is continuous, thus (26) can be rewritten as

\[ \int_{\mathbb{R}^+} G_X \circ f^{-1} d\mu = \int_{\mathbb{R}^+} G_X(t)df(t). \]

Combining with (25), the lemma is proved. \( \square \)
In the following we prove Theorem 2.1.

**Proof of Theorem 2.1.** From the definition of \( f^*(x) \), using (22) we have

\[
\rho_g[Tf^*(X)] = \rho_g[X] + \int_0^\infty G_X(t)df^*(t)
\]

\[
= \rho_g[X] + \sum_{i=1}^{n^{(1)}_{r,g}} \int_{S_X^{(1)}(q_{r,g}(i))} G_X(t)dt + \sum_{i=1}^{n^{(2)}_{r,g}} \int_{S_X^{(1)}(\hat{q}_{r,g}(i))} G_X(t)dt \\
+ \sum_{i=1}^{m^{(1)}_{r,g}} \int_{S_X^{(1)}(t_{r,g}(i))} G_X(t)dt + \sum_{i=1}^{m^{(2)}_{r,g}} \int_{S_X^{(1)}(\hat{t}_{r,g}(i))} G_X(t)dt
\]

\[
= \rho_g[X] + \sum_{i=1}^{n^{(1)}_{r,g}} \int_{(S_X^{(1)}(q_{r,g}(i)), S_X^{(1)}(\hat{q}_{r,g}(i)))} G_X(t)dt \\
+ \sum_{i=1}^{n^{(2)}_{r,g}} \int_{(S_X^{(1)}(t_{r,g}(i)), S_X^{(1)}(\hat{t}_{r,g}(i)))} G_X(t)dt.
\]

Then applying the decomposition of \( \Gamma_{r,g}(X) \), we have

\[
\rho_g[Tf^*(X)] = \rho_g[X] + \int_{\{r(S_X(t)) - g(S_X(t)) < 0\}/N_1} G(S_X(t))dt
\]

\[
= \rho_g[X] + \int_{r(S_X(t)) - g(S_X(t)) < 0} G(S_X(t))dt \\
= \int_{r(S_X(t)) - g(S_X(t)) < 0} g(S_X(t))dt + \int_{r(S_X(t)) - g(S_X(t)) \geq 0} r(S_X(t))dt \\
= \int_0^\infty \min\{r(S_X(t)), g(S_X(t))\}dt.
\]

Thus (8) is proved.

On the other hand, since both \( f(x) \) and \( I_f(x) \) are increasing functions, we can derive that for any \( 0 \leq t < s \) we have \( 0 \leq f(s) - f(t) \leq s - t \). Thus for any \( f \in F \),

\[
\int_{R^+} G_X(t)df(t) \geq \int_{\{G_X(t) \leq 0\} \cap R^+} G_X(t)dt.
\]
Then applying (22) we can obtain that
\[
\rho_g[T_f(X)] \geq \rho_g[X] + \int_{\{G_X(t) \leq 0\} \cap \mathbb{R}^+} [G_X(t)]dt
\]
\[
= \int_{r(S_X(t)) - g(S_X(t)) \leq 0} r(S_X(t))dt + \int_{r(S_X(t)) - g(S_X(t)) > 0} g(S_X(t))dt
\]
\[
= \int_0^\infty \min\{r(S_X(t)), g(S_X(t))\}dt
\]
\[
= \rho_g[T_f^*(X)].
\]

Then \(f^*\) in (9) is optimal. Theorem 2.1 is proved. \(\square\)

**Proof of Lemma 2.1.** Note that
\[
\mu_r(f^*(X)) = \int_{r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t))dt > \pi.
\]

For simplicity, we define \(G(a) = \int_{(1+a)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t))dt\). Then \(G(a), a \geq 0\) is a decreasing function and \(G(0) = \mu_r(f^*(X)) > \pi, G(\infty) = 0\). For any sequence \(a_n \downarrow a\), since

\[
{t : (1 + a)r(S_X(t)) - g(S_X(t)) < 0} \supseteq {t : (1 + a_n)r(S_X(t)) - g(S_X(t)) < 0}
\]
\[
= {t : (1 + a)r(S_X(t)) - g(S_X(t)) < (a - a_n)r(S_X(t))},
\]

then we have
\[
\lim_{a_n \downarrow a} \{t : (1 + a_n)r(S_X(t)) - g(S_X(t)) < 0\} = \{t : (1 + a)r(S_X(t)) - g(S_X(t)) < 0\},
\]
which implies that \(\lim_{a_n \downarrow a} G(a_n) = G(a)\). Therefore \(G(a)\) is a right continuous function.

(1)Consider the case that there exists \(a > 0\) such that \(G(a) = \pi\). Then \(\theta\) can be represented as
\[
\theta = \inf\{a : \int_{(1+a)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t))dt = \pi\}.
\]
From the definition of \(\theta\), there is a sequence \(\theta_n \downarrow \theta\) and \(\int_{(1+\theta_n)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t))dt = \pi\). By the right continuity of \(G(a)\), we have \(G(\theta) = \pi\). Thus (10) is proved.
(2) Consider the case that \( G(a) = \pi, a \geq 0 \) has no solution. Then \( \theta \) can be represented as

\[
\theta = \inf \{ a : \int_{(1+a)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t)) dt < \pi \}.
\]

From the definition of \( \theta \), there is a sequence \( \theta_n \downarrow \theta \) and \( \int_{(1+\theta_n)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t)) dt < \pi \). By the right continuity of \( G(a) \), we have \( G(\theta) = \lim_{\theta_n \downarrow \theta} G(\theta_n) \leq \pi \). Combining with the fact that \( G(a) = \pi, a \geq 0 \) has no solution, we have \( G(\theta) < \pi \). Meanwhile, \( G(\theta-) \geq \pi \) can be obtained immediately from the definition of \( \theta \). Furthermore, \( G(\theta-) = \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) \leq 0} r(S_X(t)) dt \) follows and (11) is proved. \( \square \)

**Proof of Theorem 2.2.** We have

\[
\begin{align*}
\rho_g[T_{f^*}(X)] &= \rho_g[X] + \int_0^\infty G_X(t)d\pi_1(t) \\
&= \rho_g[X] + \int_{\{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0\}} G_X(t)d\pi_1(t) \\
&\quad + \int_{\{(1+\theta)r(S_X(t)) - g(S_X(t)) = 0\}} G_X(t)d\pi_2(t) \\
&= \rho_g[X] + \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} G_X(t)d\pi_1(t) + \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) = 0} G_X(t)d\pi_2(t) \\
&= \rho_g[X] + \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} G_X(t)dt - \frac{\pi - \pi_1}{\pi_2 - \pi_1} \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) = 0} G_X(t)dt \\
&= \rho_g[X] + \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} G_X(t)dt - \theta(\pi - \pi_1).
\end{align*}
\]
For any \( f \in \mathcal{F} \) satisfying \( \mu_r(f(X)) \leq \pi \), using Lemma 4.1 we have

\[
\rho_g'[T_f^{**}](X) - \rho_g[T_f(X)] = \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} G_X(t) dt - \theta(\pi - \pi_1) - \int_{\mathbb{R}+} G_X(t) df(t)
\]

\[
= \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} G_X(t) dt - \theta(\pi - \pi_1) - \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} G_X(t) df(t) - \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) \geq 0} G_X(t) df(t)
\]

Following the inequality

\[
\int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} G_X(t) df(t) \leq -\theta \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t)) df(t),
\]

we get

\[
\rho_g'[T_f^{**}](X) - \rho_g[T_f(X)] \leq -\theta \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t)) df(t) - \theta(\pi - \pi_1)
\]

\[
+ \theta \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) = 0} r(S_X(t)) df(t) + \theta \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) > 0} r(S_X(t)) df(t)
\]

\[
= -\theta \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t)) dt + \theta \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} r(S_X(t)) df(t) - \theta(\pi - \pi_1)
\]

\[
+ \theta \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) = 0} r(S_X(t)) df(t) + \theta \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) > 0} r(S_X(t)) df(t)
\]

\[
= -\theta \{\pi_1 + (\pi - \pi_1)\} + \theta \int_0^\infty r(S_X(t)) df(t)
\]

\[
= \theta(\mu_r(f(X)) - \pi) \leq 0.
\]

Theorem 2.2 is proved.

**Proof of Theorem 2.3.** It is obvious that \( f^{**} \in \mathcal{F} \). We only need to prove

\[
\rho_g'[T_f^{**}](X) = \rho_g[X] + \int_{(1+\theta)r(S_X(t)) - g(S_X(t)) < 0} G_X(t) dt - \theta(\pi - \pi_1).
\]
Since

\[ \pi = \mu(T_{f^{***}}(X)) = \int_0^\infty r(S_X(t))df^{***}(t) \]

\[ = \int_{\Gamma_{(1+\theta)r,g}(X)} r(S_X(t))dt + \int_{\Gamma_0^{(1+\theta)r,g}(X)} r(S_X(t))df^{***}(t) \]

\[ = \pi_1 + \int_{\Gamma_0^{(1+\theta)r,g}(X)} r(S_X(t))df^{***}(t), \]

then \( \int_{\Gamma_0^{(1+\theta)r,g}(X)} r(S_X(t))df^{***}(t) = \pi - \pi_1. \) Thus we have

\[ \rho_g[T_{f^{***}}(X)] = \rho_g[X] + \int_0^\infty G_X(t)df^{***}(t) \]

\[ = \rho_g[X] + \int_{\Gamma_{(1+\theta)r,g}(X)} G_X(t)dt + \int_{\Gamma_0^{(1+\theta)r,g}(X)} G_X(t)df^{***}(t) \]

\[ = \rho_g[X] + \int_{\Gamma_{(1+\theta)r,g}(X)} G_X(t)dt - \theta \int_{\Gamma_0^{(1+\theta)r,g}(X)} r(S_X(t))df^{***}(t) \]

\[ = \rho_g[X] + \int_{\Gamma_{(1+\theta)r,g}(X)} G_X(t)dt - \theta(\pi - \pi_1). \]

Theorem 2.3 is proved. \( \square \)

4.3 Proofs of Corollaries 3.1-3.2

As proved in Theorem 2.1, ignoring countable points of \( \Gamma_{r,g}(X) \) will not change both the form of optimal ceded loss function and the minimum risk, thus we will ignore the countable points when we consider \( \Gamma_{r,g}(X), \Gamma_{(1+\theta)r,g}(X) \) and \( \Gamma_0^{(1+\theta)r,g}(X). \)

Proof of Corollary 3.1

(1) Consider the expected premium principle.

(a) In the case \( \frac{1}{1+\beta} > S_X(0), \Gamma_{r,g}(X) = (0, S_X^{-1}(\alpha)) \) and \( \Gamma_0^{r,g}(X) = \emptyset; \) In the case \( \alpha \leq \frac{1}{1+\beta} \leq S_X(0), \Gamma_{r,g}(X) = (S_X^{-1}(\frac{1}{1+\beta}), S_X^{-1}(\alpha)) \) and \( \Gamma_0^{r,g}(X) = (S_X^{-1}(\frac{1}{1+\beta}), S_X^{(-1)}(\frac{1}{1+\beta})). \)

In the case \( \frac{1}{1+\beta} < \alpha, \Gamma_{r,g}(X) = \Gamma_0^{r,g}(X) = \emptyset. \) For the above three cases,

\[ \Gamma_{r,g}(X) \cup \Gamma_0^{r,g}(X) = (S_X^{-1}(\frac{1}{1+\beta}), S_X^{-1}(\alpha)). \]

Applying Theorem 2.1, \( f^*_{E, VaR}(x) \) is one optimal solution of optimization problem (16).
(b) If $E[f_{E,V,aR}^*(X)] \leq B$, $f_{E,V,aR}^*(x)$ is also a solution of (17). Otherwise, $E[f^*(X)] > B$, then we can find a unique $\theta > 0$ by Lemma 2.1. Write $1 + \tilde{\beta} = (1 + \theta)(1 + \beta)$ and we get
\[
\Gamma_{(1+\theta)r,g}(X) = (S_X^{-1}(1\frac{1}{1+\beta}), S_X^{-1}(\alpha)) \text{ and } \Gamma^0_{(1+\theta)r,g}(X) = (S_X^{-1}(1\frac{1}{1+\beta}), S_X^{(-1)}(1\frac{1}{1+\beta})).
\]
Thus from Theorem 2.2 we get that $f_{E,V,aR}^{**}(x)$ is one optimal solution, where $\lambda_1$ is determined by $E(f_{E,V,aR}^{**}(X)) = B$. Similarly, applying Theorem 2.3 we can find that for $d_1^* \in [S_X^{-1}(\frac{1}{1+\beta}), S_X^{(-1)}(\frac{1}{1+\beta})]$ with $\int_{d_1^*}^{S_X^{-1}(\alpha)} S_X(t) dt = B$, $f_{E,V,aR}^{***}(x)$ is another optimal solution of (17).

(2) Consider Wang's premium principle.

(a) For the optimization problem (16), $\Gamma_{r,g}^-(X) = (0, S_X^{-1}(\alpha))$, then we know from Theorem 2.1 that $f_{W,V,aR}^*(x) = x \wedge S_X^{-1}(\alpha)$ is one optimal solution of (16).

(b) If $\mu_r(f_{W,V,aR}^*(X)) \leq \pi$, $f_{W,V,aR}^*(x)$ is one solution of the optimization problem (16). Otherwise, we can find a $\theta > 0$ by Lemma 2.1. The unique solution $q_w^*$ of $1 = (1+\theta)r(x)$ satisfies that $\int_{S_{X}^{-1}(q_w^*)}^{S_{X}^{-1}(\alpha)} S_{X}(t) dt \leq B \leq \int_{S_{X}^{-1}(q_w^*)}^{S_{X}^{-1}(\alpha)} S_{X}(t) dt$. Then by Theorem 2.3, we obtain a $d_2^* \in [S_{X}^{-1}(q_w^*), S_{X}^{(-1)}(q_w^*)]$ such that $f_{W,V,aR}^{**}(x) = \min\{x - d_2^* +, S_{X}^{-1}(\alpha) - d_2^*\}$ is one optimal solution of the optimization problem (16), where $d_2^*$ is determined by $\mu_r(f_{W,V,aR}^{**}(X)) = \pi$. □

Proof of Corollary 3.2

(1) For the optimization problem (18), from the definition of $q_w^*$, we have $\Gamma_{r,g}^-(X) = (S_{X}^{(-1)}(q_w^*), \infty)$ and $\Gamma_{r,g}^0(X) \supseteq (S_{X}^{-1}(q_w^*), S_{X}^{(-1)}(q_w^*))$. Thus by Theorem 2.1, we know that
\[
f_{E,Conc}^*(x) = (x - S_{X}^{-1}(q_w^*))_+
\]
is one solution of the optimization problem (18).

(2) If $E[f_{E,Conc}^*(X)] \leq B$, it is obvious that $f_{E,Conc}^*(x)$ is one solution of the optimization problem (18). Otherwise, we can find a unique $\theta > 0$ by Lemma 2.1. The unique
nonzero solution $\tilde{q}_3^*$ of $g(x) = (1 + \theta)(1 + \beta)x$ satisfies that $\int_{S_X^{-1}(\tilde{q}_3^*)}^{\infty} S_X(t) dt \leq B \leq \int_{S_X^{-1}(\tilde{q}_3^*)}^{\infty} S_X(t) dt$. Then by Theorem 2.3, there exists a $d_3^* \in [S_X^{-1}(\tilde{q}_3^*), S_X^{-1}(\tilde{q}_3^*)]$ such that
\[
E[f_{E, Conc}^*(x)] = (x - d_3^*)^+
\]
is one optimal solution, where $d_3^*$ is determined by $E[f_{E, Conc}^*(X)] = B$. □

5 Conclusions

This paper discusses the optimal reinsurance model when the insurer’s risk is measured by distortion risk measure and the reinsurance premium is determined by a general premium principle including expected premium principle and Wang’s premium principle as its special cases. We present a new method to discuss the optimal problems, and the optimal reinsurance strategies are obtained. Our theoretical results show that the optimal reinsurance strategy is a combination of limited stop-loss and quota share reinsurance. Based on our methods, it is easier to explain the optimal reinsurance treaty in the view of a balance between the insurer’s risk attitude and the reinsurance premium principle.

Acknowledgment. We thank Dr. Yichun Chi for his helpful comments. Yang’s research was supported by the Key Program of National Natural Science Foundation of China (Grant No. 11131002) and the National Basic Research Program (973 Program) of China (2007CB814905).

References


